The Ground Rules (and other rules) of Proof Writing

The Ground Rules of Proof Writing

Every statement in a proof must be one of the following:

1) the **Definition** of a variable, OR

2) a **Supposition** or an **Assumption**, OR

3) a **Deduction** that a statement is TRUE from the statements *that have come before it*

(either appearing explicitly in the proof or from theorems proven earlier) *and the rules of logic*.

A Deduction must be accompanied by a reason justifying it.

Most deductions will begin “Therefore,” “∴,” or with some other term: “So,” “Thus,” etc.

All other statements in a proof are descriptive COMMENTS.

All comments must be written in brackets: [Comments go here!]

The Definition of a Variable

In this class, the initial definition of a variable should be accomplished using the word "Let", as in:

“Let n be a positive even integer.”

(This means “Let us agree that n will represent an arbitrary but particular positive even integer.”)

Sometimes is it desirable to define the variable initially (using "Let") as belonging to a larger class and then to restrict the domain of the variable further, using the words "Suppose" or "Assume", as in:

“Let n be a positive integer.
Suppose that n is even.” OR “Let n be a positive integer.
Assume that n is even.”

Let is ONLY used for the *initial* definition of a variable. ***Let* is not used for the secondary domain restriction.**
For secondary restriction of the domain, “Suppose” or “Assume” should be used.

Thus, the following is incorrect to say in a proof: “Let n be a positive integer.
Let n be even.” ← **NOT GOOD**

Suppose and *Assume* are used to make assumptions regarding the situation discussed, thereby establishing a restricted context in which the argument proceeds. This is a powerful action, sometimes diverting the argument into impossible situations (as in a Proof-by-Contradiction).

The author of the textbook, Susanna Epp, uses *Suppose* in the initial definition of the variable
(as in using “Suppose n is a positive integer” to define variable n initially).
This is overkill and should be discouraged.

ENCOURAGED	DISCOURAGED
Let n be a positive integer. or Let positive integer n be given.	Suppose n is a positive integer. and Suppose positive integer n is given.

The only two phrases allowed (in this course) to be used in an Existential Statement are

"**There exists** *(variable definition)* **such that** *(predicate in terms of the variable)*"

and "**For some** *(variable definition)***,** *(predicate in terms of the variable)*" .

[These phrases are sometimes reversed:
"*(predicate in terms of a variable),* **for some** *(definition of that variable)* . "]

Example: The following three proof fragments correctly define variable k in an existential statement:

"Let n be an even integer.
By definition of even, **there exists** an integer k **such that** n = 2k ."

"Let n be an even integer.
Therefore, **for some** integer k**,** n = 2k, by definition of 'even.' "

"Let n be an even integer.
Therefore, n = 2k, **for some** integer k**,** by definition of 'even.' "

The word '**some**' here is critical and it cannot be omitted!

To say, "For integer k, n = 2k" is ambiguous. ← **INCORRECT!**

Does it mean "For **every** integer k, n = 2k " or does it mean "For **some** integer k, n = 2k " ?

("For **every** integer k, . . . " and "For **some** integer k, . . ." have vastly different meanings!)

It is incorrect to use *where* to assert existence. *Where* cannot be used to claim the existence of something.

Thus, it is <u>incorrect</u> to say:

"Let n be an even integer.
By definition of 'even', n = 2k, <u>where k is an integer</u>. " ← **INCORRECT!**

Where can be used only to define a variable to represent something which is known to exist already.

The word *Where* can be used in the initial definition of a variable, when it comes immediately after the initial use of that variable and in the same sentence containing the variable's first use.

(Variables m and n have already been defined as representing integers and it has been established that n is odd and that m = 6n + 14 .)

<u>CORRECT Use of *Where* to define t</u>:

$\therefore$ m = 2 (3n + 7) by rules of algebra.
$\therefore$ m = 2 t , **where** t = 3n + 7 .

(The number (3n + 7) is already known to exist and the wording defines t as representing that number. No claim of existence is being made.)

<u>A MORE FOMAL WORDING (Not Using *Where*):</u>

$\therefore$ m = 2 (3n + 7) , by rules of algebra.

Let t = 3n + 7 .

$\therefore$ m = 2 t , by substitution.

Proving that an Equation or an Inequality is True

Recall that every statement in a proof is either a Definition, an Assumption, or a Deduction.
When a particular statement is not a definition and is also not an assumption, then it must be interpreted as being a deduction.
The Rule of Deduction Statements: Every deduction statement must be seen to be true by the rules of logic **and by what has been proved before that statement is written**, either by what was established earlier in the proof or by a theorem that has already been proved.

Now, there are two methods by which students **attempt** in their proofs to prove that a particular equation is true.

Method 1: 1) Write down the equation to be proved.

2) Write down another equation by replacing an expression on one side of the equation with an equivalent but simpler expression.

3) Continue in this manner until the same expressions is on both sides of the final equation.

The following unacceptable proof uses this method.

To Prove: $\left(\tan^2 x\right)\left(\cos^3 x\right)\sec^2 x \quad = \quad \sec x\ \sin^2 x$.

Proof: $\left(\tan^2 x\right)\left(\cos^3 x\right)\sec^2 x \quad = \quad \sec x\ \sin^2 x$.

$$\left(\tan^2 x\right)\left(\cos^3 x\right)\left(\frac{1}{\cos^2 x}\right) = \frac{1}{\cos x}\sin^2 x \qquad \text{since} \qquad \sec x = \frac{1}{\cos x}\ .$$

$$\left(\tan^2 x\right)\left(\cos x\right) = \frac{1}{\cos x}\sin^2 x \qquad \text{by Rules of Algebra .}$$

$$\left(\frac{\sin^2 x}{\cos^2 x}\right)\left(\cos x\right) = \frac{\sin^2 x}{\cos x} \qquad \text{since} \qquad \tan x = \frac{\sin x}{\cos x}\ .$$

$$\frac{\sin^2 x}{\cos x} = \frac{\sin^2 x}{\cos x} \qquad \text{CHECK! So, the given equation has been proved True.}$$

Done.

The proof above is UNACCEPTABLE!

Why is this Method 1 proof UNACCEPTABLE???? Because it fails to obey the Rule of Deduction Statements presented above.

The very first statement, $\left(\tan^2 x\right)\left(\cos^3 x\right)\sec^2 x \quad = \quad \sec x\ \sin^2 x$, is a deduction statement (because it is neither a definition of a variable nor an assumption. As a deduction, it must be seen to be true by what has come before it was written. But, in the Method 1 proof method, the first statement is presented without a justification telling how it follows from previous statements. In fact, it does not follow from **previous** statements. Instead, it seems to follow from **later** statements, which is not a valid proof method in Logic.

The second method by which students **attempt and succeed** in their proofs to prove that a particular equation is true is as follows:

Method 2: 1) Write the expression from one side of the equation and set it equal to another expression that the the first expression is seen to be equivalent to by a particular identity, by the rules of Algebra, or by a principle from Calculus.

2) Write the new second expression and set it equal to a third expression that the second expression is seen to be equivalent to by a particular identity, by the rules of Algebra, or by a principle from Calculus.

3) Continue in this manner, always setting each new equivalent expression equal to another equivalent expression, until you arrive at the expression on the opposite side of the original equation.

By transitivity, then, the expression on one side of the equation to be proved has been proved to be equivalent to the expression on the other side of the equation to be proved.

The following acceptable and correct proof uses this method.

To Prove: $\left(\tan^2 x\right)\left(\cos^3 x\right)\sec^2 x \quad = \quad \sec x \ \sin^2 x$.

Proof: [We start with the left-hand side of the equation and arrive at the right-hand side.]

$$\left(\tan^2 x\right)\left(\cos^3 x\right)\sec^2 x \quad = \quad \left(\tan^2 x\right)\left(\cos^3 x\right)\left(\frac{1}{\cos^2 x}\right) \qquad \text{since} \qquad \sec x \quad = \quad \frac{1}{\cos x} \ .$$

$$\left(\tan^2 x\right)\left(\cos^3 x\right)\left(\frac{1}{\cos^2 x}\right) \quad = \quad \left(\tan^2 x\right)\left(\cos x\right) \qquad \text{by Rules of Algebra.}$$

$$\left(\tan^2 x\right)\left(\cos x\right) \quad = \quad \left(\frac{\sin^2 x}{\cos^2 x}\right)\left(\cos x\right) \qquad \text{since} \qquad \tan x \quad = \quad \frac{\sin x}{\cos x} \ .$$

$$\left(\frac{\sin^2 x}{\cos^2 x}\right)\left(\cos x\right) \quad = \quad \frac{1}{\cos x}\ \sin^2 x \qquad \text{by Rules of Algebra.}$$

$$\frac{1}{\cos x}\ \sin^2 x \quad = \quad \sec x \ \sin^2 x \qquad \text{since} \qquad \sec x \quad = \quad \frac{1}{\cos x} \ .$$

Therefore, $\left(\tan^2 x\right)\left(\cos^3 x\right)\sec^2 x \quad = \quad \sec x \ \sin^2 x$, by transitivity of equality.

Done.

This presentation can be simplified by writing one long sentence using subordinate clauses, as in saying: "This equals that, which equals that, which equals that, which equals ....."
The above proof is written with this simpler format.

To Prove: $\left(\tan^2 x\right)\left(\cos^3 x\right)\sec^2 x \quad = \quad \sec x \; \sin^2 x$.

Proof: [We start with the left-hand side of the equation and arrive at the right-hand side.]

$$\left(\tan^2 x\right)\left(\cos^3 x\right)\sec^2 x \quad = \quad \left(\tan^2 x\right)\left(\cos^3 x\right)\left(\frac{1}{\cos^2 x}\right) \quad \text{since} \quad \sec x \; = \; \frac{1}{\cos x} \; ,$$

$$= \quad \left(\tan^2 x\right)\left(\cos x\right) \qquad \text{by Rules of Algebra,}$$

$$= \quad \left(\frac{\sin^2 x}{\cos^2 x}\right)\left(\cos x\right) \qquad \text{since} \quad \tan x \; = \; \frac{\sin x}{\cos x} \; .$$

$$= \quad \frac{1}{\cos x}\; \sin^2 x \qquad \text{by Rules of Algebra,}$$

$$= \quad \sec x \; \sin^2 x \qquad \text{since} \quad \sec x \; = \; \frac{1}{\cos x} \; .$$

Therefore, $\left(\tan^2 x\right)\left(\cos^3 x\right)\sec^2 x \quad = \quad \sec x \; \sin^2 x$, by transitivity of equality.

Done.

The reason that this Method 2 proof is ACCEPTABLE and correct is because every deduction statement is true proved true based on what had already been established to that point.

This same issue arises when you are proving an inequality. Remember that you must always start with KNOWN FACTS and then derive intermediate facts, finally establishing the truth of the inequality with deduction statements that are all seen to be true by what has come before. For example:

To Prove: $\sqrt{2} \; < \; 19/13$

Acceptable Proof:

$(13^2) \times 2 \; = \; 338 \; < \; 361 \; = \; 19^2$,

by Rules of Algebra.

$\therefore \; 2 \; < \; (19^2) \; / \; (13^2)$

[from dividing both sides by (13^2)

$\therefore \sqrt{2} \; < \; 19/13$.

[from taking the square root of both sides of the inequality.]

THIS PROOF IS CORRECT!

To Prove: $\sqrt{2} \; < \; 19/13$

A Not Acceptable proof:

$\sqrt{2} \; < \; 19/13$

$13\sqrt{2} \; < \; 19$ [from multiplying both sides by 13].

$\left(13^2\right)\times 2 \; < \; 19^2$ [from squaring both sides}.

$\therefore \; 338 \; < \; 361$. Check!! Done.

THIS PROOF IS NOT CORRECT!!